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Geometric Hardy and Hardy–Sobolev inequalities on Heisenberg groups

Michael Ruzhansky

Department of Mathematics, Ghent University
Belgium and School of Mathematical Sciences
Queen Mary University of London, UK
Michael.Ruzhansky@ugent.be

Bolys Sabitbek

Institute of Mathematics and Mathematical Modeling
125 Pushkin Street, Almaty 050010, Kazakhstan
Al-Farabi Kazakh National University
71 al-Farabi Ave., Almaty 050040, Kazakhstan
b.sabitbek@math.kz

Durvudkhan Suragan*

Department of Mathematics
Nazarbayev University, 53 Kabanbay Batyr Ave.
Astana 010000, Kazakhstan
durvudkhan.suragan@nu.edu.kz

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In this paper, we present geometric Hardy inequalities for the sub-Laplacian in half-spaces of stratified groups. As a consequence, we obtain the following geometric Hardy inequality in a half-space of the Heisenberg group with a sharp constant:

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\operatorname{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \quad p > 1,$$

which solves a conjecture in the paper [S. Larson, Geometric Hardy inequalities for the sub-elliptic Laplacian on convex domain in the Heisenberg group, *Bull. Math. Sci.*

*Corresponding author.

6 (2016) 335–352]. Here,

$$\mathcal{W}(\xi) = \left(\sum_{i=1}^{n} \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2 \right)^{\frac{1}{2}}$$

is the angle function. Also, we obtain a version of the Hardy–Sobolev inequality in a half-space of the Heisenberg group:

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\operatorname{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{1}{p^*}},$$

where $\operatorname{dist}(\xi, \partial\mathbb{H}^+)$ is the Euclidean distance to the boundary, $p^* := Qp/(Q-p)$, and $2 \leq p < Q$. For $p = 2$, this gives the Hardy–Sobolev–Maz'ya inequality on the Heisenberg group.

Keywords: Stratified groups; Heisenberg group; geometric Hardy inequality; half-space; sharp constant.

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1. Introduction

Let us recall the Hardy inequality in the half-space of $\mathbb{R}^n$

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx, \quad p > 1, \tag{1.1}$$

for every function $u \in C_0^\infty(\mathbb{R}^n_+)$, where ∇ is the usual Euclidean gradient and $\mathbb{R}^n_+ := \{(x', x_n) | x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}, n \in \mathbb{N}$. There is a number of studies related to inequality (1.1), see e.g. [1, 2, 5, 12].

Filippas *et al.* in [6] established the Hardy–Sobolev inequality in the following form:

$$\left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx - \left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \right)^{\frac{1}{p}} \geq C \left(\int_{\mathbb{R}^n_+} |u|^{p^*} dx \right)^{\frac{1}{p^*}}, \tag{1.2}$$

for all function $u \in C_0^\infty(\mathbb{R}^n_+)$, where $p^* = \frac{np}{n-p}$ and $2 \leq p < n$. For a different proof of this inequality, see Frank and Loss [8] as well as Psaradakis [13].

A Hardy inequality for a half-space of the Heisenberg group was shown by Luan and Young [11] in the form

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi. \tag{1.3}$$

An alternative proof of this inequality was given by Larson in [10], where the author generalized it to any half-space of the Heisenberg group

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4} \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2}{\operatorname{dist}(\xi, \partial\mathbb{H}^+)^2} |u|^2 d\xi,$$

where X_i and Y_i (for $i = 1, \ldots, n$) are left-invariant vector fields on the Heisenberg group, ν is the Riemannian outer unit normal (see [9]) to the boundary. In the same paper the following L^p-generalization of the inequality was proved

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n |\langle X_i(\xi), \nu\rangle|^p + |\langle Y_i(\xi), \nu\rangle|^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi.$$

Note also that the authors in [15] have extended this result to general Carnot groups, see [16–19].

The main aim of this paper is to improve the L^p-version of the geometric Hardy inquality for the sub-Laplacian in the half-spaces of stratified groups (Carnot groups), where the obtained inequality will be a natural extension of the inequality derived by the authors in [10, 15] on Heisenberg and stratified groups, respectively. In particular, we prove an inequality conjectured in [10]. Moreover, we obtain a version of the Hardy–Sobolev inequality in the setting of the Heisenberg group.

The main results of this paper are as follows:

- **Geometric L^p-Hardy inequality on $\mathbb{G}^+$:** Let $\mathbb{G}^+ := \{x \in \mathbb{G} : \langle x, \nu\rangle > d\}$ be a half-space of a stratified group $\mathbb{G}$. Then for all $u \in C_0^\infty(\mathbb{G}^+)$, $\beta \in \mathbb{R}$ and $p > 1$, we have
$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{\mathcal{W}(x)^p}{\mathrm{dist}(x, \partial\mathbb{G}^+)^p} |u|^p dx$$
$$+ \beta \int_{\mathbb{G}^+} \frac{\mathcal{L}_p(\mathrm{dist}(x, \partial\mathbb{G}^+))}{\mathrm{dist}(x, \partial\mathbb{G}^+)^{p-1}} |u|^p dx,$$
where $\mathcal{W}(x) := (\sum_{i=1}^N \langle X_i(x), \nu\rangle^2)^{1/2}$, and $\mathcal{L}_p$ is the p-sub-Laplacian on $\mathbb{G}$, see (1.6).
- **Geometric L^p-Hardy inequality on $\mathbb{H}^+$:** Let $\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu\rangle > d\}$ be a half-space of the Heisenberg group. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have
$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi,$$
where $\mathcal{W}(\xi) := \left(\sum_{i=1}^n \langle X_i(\xi), \nu\rangle^2 + \langle Y_i(\xi), \nu\rangle^2\right)^{1/2}$ and the constant is sharp.
- **Geometric Hardy–Sobolev inequality on $\mathbb{H}^+$:** Let $\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu\rangle > d\}$ be a half-space of the Heisenberg group. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n+1$, there exists some $C > 0$ such that we have
$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{\frac{1}{p}} \geq C \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{1}{p^*}},$$
where $\mathrm{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu\rangle - d$ is the distance from ξ to the boundary and $p^* := Qp/(Q-p)$.

1.1. *Preliminaries on stratified groups*

Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a stratified Lie group (or a homogeneous Carnot group), with dilation structure δ_λ and Jacobian generators $X_1, \ldots, X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. Let us denote by Q the homogeneous dimension of $\mathbb{G}$. We refer to the recent books [7, 20] for extensive discussions of stratified Lie groups and their properties.

The sub-Laplacian on $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2. \tag{1.4}$$

We also recall that the standard Lebesque measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [7, Proposition 1.6.6]). Each left invariant vector field X_k has an explicit form and satisfies the divergence theorem, see e.g. [7] for the derivation of the exact formula: more precisely, we can express

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r}\sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \ldots, x^{(l-1)})\frac{\partial}{\partial x_m^{(l)}}, \tag{1.5}$$

with $x = (x', x^{(2)}, \ldots, x^{(r)})$, where r is the step of $\mathbb{G}$ and $x^{(l)} = (x_1^{(l)}, \ldots, x_{N_l}^{(l)})$ are the variables in the lth stratum, see also [7, Sec. 3.1.5] for a general presentation. The horizontal gradient is given by

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_N),$$

and the horizontal divergence is defined by

$$\mathrm{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} \cdot v.$$

The p-sub-Laplacian has the following form:

$$\mathcal{L}_p v = \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}} v|^{p-2}\nabla_{\mathbb{G}} v). \tag{1.6}$$

Let us define the half-space on the stratified group $\mathbb{G}$ as

$$\mathbb{G}^+ := \{x \in \mathbb{G} : \langle x, \nu\rangle > d\},$$

where $\nu := (\nu_1, \ldots, \nu_r)$ with $\nu_j \in \mathbb{R}^{N_j}, j = 1, \ldots, r$, is the Riemannian outer unit normal to $\partial\mathbb{G}^+$ (see [9]) and $d \in \mathbb{R}$. Let us define the so-called angle function

$$\mathcal{W}(x) := \sqrt{\sum_{i=1}^{N} \langle X_i(x), \nu\rangle^2};$$

such function was introduced by Garofalo [9] in his investigation of the horizontal Gauss map. The Euclidean distance to the boundary $\partial\mathbb{G}^+$ is denoted by $\mathrm{dist}(x, \partial\mathbb{G}^+)$ and defined as

$$\mathrm{dist}(x, \partial\mathbb{G}^+) = \langle x, \nu\rangle - d.$$

2. Geometric Hardy Inequalities

Theorem 2.1. *Let $\mathbb{G}^+$ be a half-space of a stratified group $\mathbb{G}$. Then for all $\beta \in \mathbb{R}$ and $p > 1$, we have*

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{\mathcal{W}(x)^p}{\operatorname{dist}(x, \partial\mathbb{G}^+)^p} |u|^p dx + \beta \int_{\mathbb{G}^+} \frac{\mathcal{L}_p(\operatorname{dist}(x, \partial\mathbb{G}^+))}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{p-1}} |u|^p dx, \tag{2.1}$$

for all $u \in C_0^\infty(\mathbb{G}^+)$.

Proof of Theorem 2.1. Let us begin with the divergence theorem, then we apply the Hölder inequality and the Young inequality, respectively. It follows for a vector field $V \in C^\infty(\mathbb{G}^+)$ that

$$\begin{aligned}\int_{\mathbb{G}^+} \operatorname{div}_{\mathbb{G}} V |u|^p dx &= -p \int_{\mathbb{G}^+} |u|^{p-1} \langle V, \nabla_{\mathbb{G}} |u| \rangle dx \\ &\leq p \left(\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} |u||^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}^+} |V|^{\frac{p}{p-1}} |u|^p dx \right)^{\frac{p-1}{p}} \\ &\leq \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} |u||^p dx + (p-1) \int_{\mathbb{G}^+} |V|^{\frac{p}{p-1}} |u|^p dx.\end{aligned}$$

By rearranging the above expression and using $|\nabla_{\mathbb{G}} u| \geq |\nabla_{\mathbb{G}} |u||$, we arrive at

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \geq \int_{\mathbb{G}^+} (\operatorname{div}_{\mathbb{G}} V - (p-1)|V|^{\frac{p}{p-1}}) |u|^p dx. \tag{2.2}$$

Now, we choose V in the following form:

$$V = \beta \frac{|\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial\mathbb{G}^+)|^{p-2}}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{p-1}} \nabla_{\mathbb{G}} \operatorname{dist}(x, \partial\mathbb{G}^+), \tag{2.3}$$

that is

$$|V|^{\frac{p}{p-1}} = |\beta|^{\frac{p}{p-1}} \frac{|\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial\mathbb{G}^+)|^p}{\operatorname{dist}(x, \partial\mathbb{G}^+)^p}.$$

Also, we have

$$\begin{aligned}|\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial\mathbb{G}^+)|^p &= |(X_1 \operatorname{dist}(x, \partial\mathbb{G}^+), \ldots, X_N \operatorname{dist}(x, \partial\mathbb{G}^+))|^p \\ &= |(\langle X_1(x), \nu \rangle, \ldots, \langle X_N(x), \nu \rangle)|^p \\ &= \left(\sum_{i=1}^N \langle X_i(x), \nu \rangle^2 \right)^{\frac{p}{2}} = \mathcal{W}(x)^p.\end{aligned}$$

Indeed, let us show that $\langle X_i(x), \nu\rangle = X_i\langle x, \nu\rangle$:

$$X_i(x) = (\overbrace{0, \ldots, 1}^{i}, \ldots, 0, \underbrace{a_{i,1}^{(2)}(x'), \ldots, a_{i,N_2}^{(2)}(x')}_{N_2}, \ldots,$$

$$\underbrace{a_{i,1}^{(r)}(x', x^{(2)}, \ldots, x^{(r-1)}), \ldots a_{i,N_r}^{(r)}(x', x^{(2)}, \ldots, x^{(r-1)})}_{N_r}),$$

$$\langle X_i(x), \nu\rangle = \nu_i' + \sum_{l=2}^{r}\sum_{m=1}^{N_l} a_{i,m}^{(l)}(x', x^{(2)}, \ldots, x^{(r-1)})\nu_m^{(l)},$$

and

$$\langle x, \nu\rangle = \sum_{k=1}^{N} x_k'\nu_k' + \sum_{l=2}^{r}\sum_{m=1}^{N_l} x_m^{(l)}\nu_m^{(l)},$$

$$X_i\langle x, \nu\rangle = \nu_i' + \sum_{l=2}^{r}\sum_{m=1}^{N_l} a_{i,m}^{(l)}(x', x^{(2)}, \ldots, x^{(r-1)})\nu_m^{(l)}.$$

A direct calculation shows that

$$\begin{aligned}\operatorname{div}_{\mathbb{G}} V &= \beta\frac{\nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+)|^{p-2}\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+))}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{p-1}} \\ &\quad - \beta(p-1)\frac{\begin{array}{c}|\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+)|^{p-2}\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+) \\ \operatorname{dist}(x, \partial\mathbb{G}^+)^{p-2}\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+)\end{array}}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{2(p-1)}} \\ &= \beta\frac{\mathcal{L}_p(\operatorname{dist}(x, \partial\mathbb{G}^+))}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{p-1}} - \beta(p-1)\frac{|\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+)|^p}{\operatorname{dist}(x, \partial\mathbb{G}^+)^p}.\end{aligned}$$

So, we get

$$\begin{aligned}\operatorname{div}_{\mathbb{G}} V - (p-1)|V|^{\frac{p}{p-1}} &= -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta)\frac{|\nabla_{\mathbb{G}}\operatorname{dist}(x, \partial\mathbb{G}^+)|^p}{\operatorname{dist}(x, \partial\mathbb{G}^+)^p} \\ &\quad + \beta\frac{\mathcal{L}_p(\operatorname{dist}(x, \partial\mathbb{G}^+))}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{p-1}}.\end{aligned}$$

Putting the above expression into inequality (2.2), we arrive at

$$\begin{aligned}\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx &\geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta)\int_{\mathbb{G}^+} \frac{\left(\sum_{i=1}^{N}\langle X_i(x), \nu\rangle^2\right)^{\frac{p}{2}}}{\operatorname{dist}(x, \partial\mathbb{G}^+)^p}|u|^p dx \\ &\quad + \beta\int_{\mathbb{G}^+} \frac{\mathcal{L}_p(\operatorname{dist}(x, \partial\mathbb{G}^+))}{\operatorname{dist}(x, \partial\mathbb{G}^+)^{p-1}}|u|^p dx,\end{aligned}$$

completing the proof. ∎

2.1. *Preliminaries on the Heisenberg group*

Let us give a brief introduction of the Heisenberg group. Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \widetilde{\xi} := (z + \widetilde{z}, t + \widetilde{t} + 2\,\mathrm{Im}\langle z, \widetilde{z}\rangle),$$

where $\xi := (x, y, t) \in \mathbb{H}^n$, $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, $z := (x, y) \in \mathbb{R}^{2n}$ is identified by $\mathbb{C}^n$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form:

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t) \quad \text{for } \lambda > 0.$$

The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \quad \text{for } 1 \le i \le n,$$

$$Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \quad \text{for } 1 \le i \le n,$$

and with their (nonzero) commutator

$$[X_i, Y_i] = -4\frac{\partial}{\partial t}.$$

The horizontal gradient of $\mathbb{H}^n$ is given by

$$\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n),$$

so the sub-Laplacian on $\mathbb{H}^n$ is given by

$$\mathcal{L} := \sum_{i=1}^{n} (X_i^2 + Y_i^2).$$

Let us define the half-space of the Heisenberg group by

$$\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\},$$

where $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x, \nu_y \in \mathbb{R}^n$ and $\nu_t \in \mathbb{R}$ is the Riemannian outer unit normal to $\partial\mathbb{H}^+$ (see [9]) and $d \in \mathbb{R}$. Let us define the so-called angle function

$$\mathcal{W}(\xi) := \sqrt{\sum_{i=1}^{n} \langle X_i(\xi), \nu\rangle^2 + \langle Y_i(\xi), \nu\rangle^2}.$$

The Euclidean distance to the boundary $\partial\mathbb{H}^+$ is denoted by $\mathrm{dist}(\xi, \partial\mathbb{H}^+)$ and defined by

$$\mathrm{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d.$$

2.2. *Consequences on the Heisenberg group*

As a consequence of Theorem 2.1, we have the following inequality.

Corollary 2.2. *Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$, we have*

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\operatorname{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \tag{2.4}$$

where the constant is sharp.

Remark 2.3. Note that inequality (2.4) was conjectured in [10] which is a natural extension of inequality (1.3) in [11]. Also, the sharpness of inequality (2.4) was proved by choosing $\nu := (1, 0, \ldots, 0)$ and $d = 0$ in the paper [10].

Proof of Corollary 2.2. Let us rewrite the inequality in Theorem 2.1 in terms of the Heisenberg group as follows:

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\operatorname{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi$$
$$+ \beta \int_{\mathbb{H}^+} \frac{\mathcal{L}_p(\operatorname{dist}(\xi, \partial\mathbb{H}^+))}{\operatorname{dist}(\xi, \partial\mathbb{H}^+)^{p-1}} |u|^p d\xi.$$

In the case of the Heisenberg group, we need to show that the last term vanishes to prove Corollary 2.2. Indeed, we have

$$\begin{aligned}
\mathcal{L}_p(\operatorname{dist}(\xi, \partial\mathbb{H}^+)) &= \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}}(\langle \xi, \nu\rangle - d)|^{p-2} \nabla_{\mathbb{G}}(\langle \xi, \nu\rangle - d) \\
&= \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}}\langle \xi, \nu\rangle|^{p-2} \nabla_{\mathbb{G}}\langle \xi, \nu\rangle) \\
&= \sum_{i=1}^n (X_i(|\nabla_{\mathbb{G}}\langle \xi, \nu\rangle|^{p-2}\langle X_i(\xi), \nu\rangle) + Y_i(|\nabla_{\mathbb{G}}\langle \xi, \nu\rangle|^{p-2}\langle Y_i(\xi), \nu\rangle)) \\
&= \sum_{i=1}^n X_i \left(\left(\sum_{j=1}^n ((\langle X_j(\xi), \nu\rangle)^2 + (\langle Y_j(\xi), \nu\rangle)^2)\right)^{\frac{p-2}{2}} \langle X_i(\xi), \nu\rangle\right) \\
&\quad + \sum_{i=1}^n Y_i \left(\left(\sum_{j=1}^n ((\langle X_j(\xi), \nu\rangle)^2 + (\langle Y_j(\xi), \nu\rangle)^2)\right)^{\frac{p-2}{2}} \langle Y_i(\xi), \nu\rangle\right) \\
&= (p-2) \sum_{i=1}^n \left(\left(\sum_{j=1}^n ((\langle X_j(\xi), \nu\rangle)^2 + (\langle Y_j(\xi), \nu\rangle)^2)\right)^{\frac{p-2}{2}-1}\right. \\
&\quad \left. \times \langle Y_i(\xi), \nu\rangle (X_i \langle Y_i(\xi), \nu\rangle)\langle X_i(\xi), \nu\rangle\right)
\end{aligned}$$

$$
+ (p-2)\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}((\langle X_j(\xi),\nu\rangle)^2 + (\langle Y_j(\xi),\nu\rangle)^2)\right)^{\frac{p-2}{2}-1}\right.
$$

$$
\left.\times \langle X_i(\xi),\nu\rangle (Y_i\langle X_i(\xi),\nu\rangle)\langle Y_i(\xi),\nu\rangle\right) = 0,
$$

since

$$
\begin{aligned}
\langle X_i(\xi),\nu\rangle &= \nu_{x,i} + 2y_i\nu_t, \quad \langle Y_i(\xi),\nu\rangle = \nu_{y,i} - 2x_i\nu_t,\\
X_i\langle X_i(\xi),\nu\rangle &= 0, \quad Y_i\langle Y_i(\xi),\nu\rangle = 0,\\
Y_i\langle X_i(\xi),\nu\rangle &= 2\nu_t, \quad X_i\langle Y_i(\xi),\nu\rangle = -2\nu_t,
\end{aligned}
$$

where $\xi := (x,y,t)$ with $x,y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x,\nu_y,\nu_t)$ with $\nu_x := (\nu_{x,1},\ldots,\nu_{x,n})$ and $\nu_y := (\nu_{y,1},\ldots,\nu_{y,n})$. Then, we have

$$
X_i(\xi) = (\underbrace{\overbrace{0,\ldots,1}^{i},\ldots,0}_{n},\underbrace{0,\ldots,0}_{n},2y_i),
$$

$$
Y_i(\xi) = (\underbrace{0,\ldots,0}_{n},\underbrace{\overbrace{0,\ldots,1}^{i},\ldots,0}_{n},-2x_i).
$$

So, we have

$$
\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta)\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\operatorname{dist}(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi.
$$

Now, we optimize by differentiating the above inequality with respect to β, so that we have

$$
\frac{p}{p-1}|\beta|^{\frac{1}{p-1}} + 1 = 0,
$$

which leads to

$$
\beta = -\left(\frac{p-1}{p}\right)^{p-1}.
$$

Using this value of β, we arrive at

$$
\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\operatorname{dist}(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi.
$$

We have finished the proof of Corollary 2.2. $\square$

3. Geometric Hardy–Sobolev Inequalities

In this section, we present the geometric Hardy–Sobolev inequality in the half space on the Heisenberg group.

3.1. *A lower estimate for the geometric Hardy type inequalities*

We start with an estimate for the remainder.

Lemma 3.1. *Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for $p \geq 2$, there exists a constant $C_p > 0$ such that*

$$E_p[u] = \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{|\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi$$
$$\geq C_p \int_{\mathbb{H}^+} |\mathrm{dist}(\xi, \partial\mathbb{H}^+)|^{p-1} |\nabla_H v|^p d\xi, \tag{3.1}$$

for all $u \in C_0^\infty(\mathbb{H}^+)$, where $\mathrm{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d$ is the distance from ξ to the boundary, $C_p = (2^{p-1}-1)^{-1}$, and $u(\xi) = \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi)$.

The Euclidean version of such a lower estimate to the Hardy inequality was established by Barbaris *et al.* [4].

Proof of Lemma 3.1. Let us begin by recalling once again the angle function, denoted by $\mathcal{W}$,

$$|\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|^p = |(X_1\langle \xi, \nu \rangle, \ldots, X_n\langle \xi, \nu \rangle, Y_1\langle \xi, \nu \rangle, \ldots, Y_n\langle \xi, \nu \rangle)|^p$$
$$= \left(\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2\right)^{\frac{p}{2}} = \mathcal{W}(\xi)^p. \tag{3.2}$$

Note that $X_i\langle \xi, \nu \rangle$ is equal to $\langle X_i(\xi), \nu \rangle$, see the proof of Theorem 2.1. This expression $|\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|^p = \mathcal{W}(\xi)^p$ will be used later. For now, we will estimate the following form:

$$E_p[u] := \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi. \tag{3.3}$$

To estimate this, we introduce the following transformation:

$$u(\xi) = \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi). \tag{3.4}$$

By inserting it into (3.3) and using (3.2), we have

$$E_p[u] = \int_{\mathbb{H}^+} \left| \frac{p-1}{p} \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} \nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+) v + \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \nabla_H v \right|^p d\xi$$
$$-\left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |\mathrm{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v|^p d\xi$$
$$\geq \int_{\mathbb{H}^+} \left| \frac{p-1}{p} \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v + \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \frac{\nabla_H v}{\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)} \right|^p |\mathcal{W}(\xi)|^p$$
$$- \left| \frac{p-1}{p} \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v \right|^p |\mathcal{W}(\xi)|^p d\xi.$$

Then for $p \geq 2$ and $A, B \in \mathbb{R}^n$, we have that

$$|A+B|^p - |A|^p \geq C_p|B|^p + p|A|^{p-2}A \cdot B,$$

where $C_p = (2^{p-1}-1)^{-1}$ (see [4, Lemma 3.3]). By taking

$$A := \frac{p-1}{p}\mathrm{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}}v \quad \text{and} \quad B := \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}}\frac{\nabla_H v}{\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)},$$

then we have the following lower estimate:

$$\begin{aligned} E_p[u] &\geq \int_{\mathbb{H}^+} |\mathcal{W}(\xi)|^p(|A+B|^p - |A|^p)d\xi \\ &\geq C_p \int_{\mathbb{H}^+} \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{p-1}|\nabla_H v|^p \frac{\mathcal{W}(\xi)^p}{|\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|^p}d\xi + \left(\frac{p-1}{p}\right)^{p-1} \\ &\quad \times \int_{\mathbb{H}^+} |\mathcal{W}(\xi)|^p|\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|^{p-2}\left(\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+) \cdot \nabla_H|v|^p\right)d\xi \\ &\geq C_p \int_{\mathbb{H}^+} \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{p-1}|\nabla_H v|^p d\xi. \end{aligned}$$

In the last line, we have used (3.2) and we dropped the last term on the right-hand side. This completes the proof of Lemma 3.1. □

3.2. *Geometric Hardy–Sobolev inequality in the half-space on* $\mathbb{H}^n$

Now, we are ready to obtain the geometric Hardy–Sobolev inequality in the half-space on the Heisenberg group $\mathbb{H}^n$.

Theorem 3.2. *Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. For $2 \leq p < Q$ with $Q = 2n+1$ there exists some $C > 0$ such that for every function $u \in C_0^\infty(\mathbb{H}^+)$ we have*

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p}|u|^p d\xi\right)^{\frac{1}{p}} \geq C\left(\int_{\mathbb{H}^+} |u|^{p^*}d\xi\right)^{\frac{1}{p^*}}, \tag{3.5}$$

where $p^* := Qp/(Q-p)$ and $\mathrm{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu\rangle - d$ is the distance from ξ to the boundary.

Remark 3.3. Note that for $p = 2$ we have the Hardy–Sobolev–Maz'ya inequality in the following form:

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \frac{1}{4}\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^2}|u|^2 d\xi\right)^{\frac{1}{2}} \geq C\left(\int_{\mathbb{H}^+} |u|^{2^*}d\xi\right)^{\frac{1}{2^*}}, \tag{3.6}$$

where $2^* := 2Q/(Q-2)$.

Proof of Theorem 3.2. Our key ingredient of proving the Hardy–Sobolev inequality in the half-space of $\mathbb{H}^n$ is the L^1-Sobolev inequality, or the Gagliardo–Nirenberg inequality. It has been established on the Heisenberg group by Baldi *et al.* in [3].

The L^1-Sobolev inequality on the Heisenberg group follows in the form:

$$c\left(\int_{\mathbb{H}^n}|g|^{\frac{Q}{Q-1}}d\xi\right)^{\frac{Q-1}{Q}}\le\int_{\mathbb{H}^n}|\nabla_H g|d\xi,$$

for some $c>0$, for every function $g\in W^{1,1}(\mathbb{H}^n)$. Now, let us set $g=|u|^{p^*(1-1/Q)}$, then we obtain

$$\begin{aligned}c\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{\frac{Q-1}{Q}}&\le\left|\frac{p(Q-1)}{Q-p}\right|\int_{\mathbb{H}^+}|u|^{\frac{Q(p-1)}{Q-p}}|\nabla_H|u||d\xi,\\&\le\left|\frac{p(Q-1)}{Q-p}\right|\int_{\mathbb{H}^+}|u|^{\frac{Qp}{Q-p}\frac{(p-1)}{p}}|\nabla_H u|d\xi,\\&=\left|\frac{p(Q-1)}{Q-p}\right|\int_{\mathbb{H}^+}|u|^{p^*(1-1/p)}|\nabla_H u|d\xi.\end{aligned}$$

We have used $|\nabla_H|u||\le|\nabla_H u|$ (see [14, Proof of Theorem 2.1]). Then, we arrive at

$$C_1\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{\frac{Q-1}{Q}}\le\int_{\mathbb{H}^+}|u|^{p^*(1-1/p)}|\nabla_H u|d\xi,\tag{3.7}$$

where $C_1:=c\left|\frac{Q-p}{p(Q-1)}\right|>0$. Let us estimate the right-hand side of inequality (3.7). Again, we use a ground transform $u(\xi)=\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}}v(\xi)$ which leads to

$$\begin{aligned}&\int_{\mathbb{H}^+}|u|^{p^*(1-1/p)}|\nabla_H u|d\xi\\&\quad=\int_{\mathbb{H}^+}|u|^{p^*(1-1/p)}\left|\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}}\nabla_H v+\frac{p-1}{p}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{-1/p}\right.\\&\qquad\left.\times\nabla_H\operatorname{dist}(\xi,\partial\mathbb{H}^+)v\right|d\xi\\&\quad\le\int_{\mathbb{H}^+}|u|^{p^*(1-1/p)}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}}|\nabla_H v|d\xi\\&\qquad+\frac{p-1}{p}\int_{\mathbb{H}^+}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{p^*(1-1/p)^2-1/p}\\&\qquad\times|\nabla_H\operatorname{dist}(\xi,\partial\mathbb{H}^+)||v|^{p^*(1-1/p)+1}d\xi\\&\quad=I_1+\frac{p-1}{p}I_2.\end{aligned}$$

In the last line we have denoted two integrals by I_1 and I_2, respectively. Also, for simplification we denote $\alpha:=p^*(1-1/p)^2+1-1/p$. First, we estimate I_2 using integrations by parts

$$
\begin{aligned}
I_2 &= \int_{\mathbb{H}^+} \operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\alpha-1}|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)||v|^{\alpha p/(p-1)}d\xi \\
&= \frac{1}{\alpha}\int_{\mathbb{H}^+}\langle\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\alpha},\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)\rangle\frac{|v|^{\alpha p/(p-1)}}{|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|}d\xi \\
&= -\frac{1}{\alpha}\int_{\mathbb{H}^+}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\alpha}\nabla_H\left(\frac{\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|v|^{\alpha p/(p-1)}}{|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|}\right)d\xi \\
&= -\frac{1}{\alpha}\int_{\mathbb{H}^+}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\alpha} \\
&\quad\times\left(\frac{\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)\nabla_H|v|^{\alpha p/(p-1)}}{|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|}\right. \\
&\quad\left.-\frac{\langle\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+),\nabla_H|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|\rangle|v|^{\alpha p/(p-1)}}{|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|^2}\right)d\xi \\
&= -\frac{1}{\alpha}\int_{\mathbb{H}^+}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\alpha}\frac{\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)\nabla_H|v|^{\alpha p/(p-1)}}{|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|}d\xi \\
&\le -\frac{p}{p-1}\int_{\mathbb{H}^+}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\alpha}|v|^{\alpha p/(p-1)-1}|\nabla_H v|d\xi \\
&= \frac{p}{p-1}\int_{\mathbb{H}^+}|u|^{p^*(1-1/p)}\operatorname{dist}(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}}|\nabla_H v|d\xi \le \frac{p}{p-1}I_1.
\end{aligned}
$$

We have used $|\nabla_H|u|| \le |\nabla_H u|$, and

$$
\mathcal{L}(\operatorname{dist}(\xi,\partial\mathbb{H}^+)) = \sum_{i=1}^{n}(X_i\langle X_i(\xi),\nu\rangle + Y_i\langle Y_i(\xi),\nu\rangle) = 0,
$$

since

$$
\begin{aligned}
&\langle X_i(\xi),\nu\rangle = \nu_{x,i} + 2y_i\nu_t, \quad \langle Y_i(\xi),\nu\rangle = \nu_{y,i} - 2x_i\nu_t, \\
&X_i\langle X_i(\xi),\nu\rangle = 0, \quad Y_i\langle Y_i(\xi),\nu\rangle = 0,
\end{aligned}
$$

where $\xi := (x,y,t)$ with $x,y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x,\nu_y,\nu_t)$ with $\nu_x := (\nu_{x,1},\ldots,\nu_{x,n})$ and $\nu_y := (\nu_{y,1},\ldots,\nu_{y,n})$. Also, we have

$$
\begin{aligned}
&\langle\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+),\nabla_H|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|\rangle \\
&\quad= \frac{2\nu_t}{|\nabla_H \operatorname{dist}(\xi,\partial\mathbb{H}^+)|} \\
&\qquad\times\big(\underbrace{(2x_1\nu_t-\nu_{y,1})(\nu_{x,1}+2y_1\nu_t)+\cdots+(2x_n\nu_t-\nu_{y,n})(\nu_{x,n}+2y_n\nu_t)}_{n} \\
&\qquad+\underbrace{(\nu_{y,1}-2x_1\nu_t)(\nu_{x,1}+2y_1\nu_t)+\cdots+(\nu_{y,n}-2x_n\nu_t)(\nu_{x,n}+2y_n\nu_t)}_{n}\big) = 0,
\end{aligned}
$$

since

$$\begin{aligned}
\nabla_H |\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)| &= (X_1 |\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|, \ldots, X_n |\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|, \\
&\quad \times Y_1 |\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|, \ldots, Y_n |\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|) \\
&= \frac{2\nu_t}{|\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+)|} \\
&\quad \times (\underbrace{2x_1\nu_t - \nu_{y,1}, \ldots, 2x_n\nu_t - \nu_{y,n}}_{n}, \underbrace{\nu_{x,1} + 2y_1\nu_t, \ldots, \nu_{x,n} + 2y_n\nu_t}_{n}),
\end{aligned}$$

and

$$\nabla_H \mathrm{dist}(\xi, \partial\mathbb{H}^+) = (\underbrace{\nu_{x,1} + 2y_1\nu_t, \ldots, \nu_{x,n} + 2y_n\nu_t}_{n}, \underbrace{\nu_{y,1} - 2x_1\nu_t, \ldots, \nu_{y,n} - 2x_n\nu_t}_{n}).$$

As we see that integral I_2 can be estimated by integral I_1. From this estimation, we know that

$$\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \leq 2I_1. \tag{3.8}$$

Now, it comes to estimate I_1 by using the Hölder inequality

$$\begin{aligned}
I_1 &= \int_{\mathbb{H}^+} \{|u|^{p^*(1-1/p)}\}\{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^{(p-1)/p} |\nabla_H v|\} d\xi \\
&\leq \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{1-1/p} \left(\int_{\mathbb{H}^+} \mathrm{dist}(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p d\xi\right)^{1/p} \\
&\leq C_p^{-1/p} \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{1-1/p} \\
&\quad \times \left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{1/p}.
\end{aligned}$$

In the last line we have used Lemma 3.1. Inserting the estimate of I_1 in (3.8), we arrive at

$$\begin{aligned}
&\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \\
&\quad \leq 2C_p^{-1/p} \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{1-1/p} \\
&\qquad \times \left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{\frac{1}{p}}.
\end{aligned}$$

Plugging the above estimate in (3.7), we have

$$C_1\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{\frac{Q-1}{Q}} \leq 2C_p^{-1/p}\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{\frac{p-1}{p}}$$

$$\times\left(\int_{\mathbb{H}^+}|\nabla_H u|^p d\xi-\left(\frac{p-1}{p}\right)^p\int_{\mathbb{H}^+}\frac{\mathcal{W}(\xi)^p}{\mathrm{dist}(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi\right)^{\frac{1}{p}}.$$

By collecting terms, we finish the proof of Theorem 3.2. □

3.3. *Consequence of Theorem* 3.2

Let us demonstrate our result in a particular case when $p=2$.

Corollary 3.4. *Let $\mathbb{H}^+ := \{\xi=(x,y,t)\in\mathbb{H}^n | t>0\}$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u\in C_0^\infty(\mathbb{H}^+)$ taking $d=0$, we have*

$$\left(\int_{\mathbb{H}^+}|\nabla_H u|^2 d\xi-\int_{\mathbb{H}^+}\frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi\right)^{\frac{1}{2}} \geq C\left(\int_{\mathbb{H}^+}|u|^{2^*}d\xi\right)^{\frac{1}{2^*}}, \tag{3.9}$$

where $2^* := 2Q/(Q-2), Q=2n+2$, with $C>0$ independent of u.

Proof of Corollary 3.4. We have the following left-invariant vector fields:

$$X_i=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t} \quad \text{and} \quad Y_i=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t},$$

with the commutator

$$[X_i,Y_i]=-4\frac{\partial}{\partial t}.$$

Then for $\xi=(x_1,\ldots,x_n,y_1,\ldots,y_n,t)$ and $\nu=(\overbrace{0,\ldots,0}^{n},\overbrace{0,\ldots,0}^{n},1)$, we get

$$\langle X_i(\xi),\nu\rangle = 2y_i \quad \text{and} \quad \langle Y_i(\xi),\nu\rangle=-2x_i,$$

where

$$X_i(\xi)=(\overbrace{0,\ldots,1}^{i},\ldots,0,\overbrace{0,\ldots,0}^{n},2y_i),$$

$$Y_i(\xi)=(\overbrace{0,\ldots,0}^{n},\overbrace{0,\ldots,1}^{i},\ldots,0,-2x_i).$$

Thus, we arrive at

$$\frac{\mathcal{W}(\xi)^2}{\mathrm{dist}(\xi,\partial\mathbb{H}^+)^2}=4\frac{|x|^2+|y|^2}{t^2}. \tag{3.10}$$

Plugging the above expression into inequality (3.6), we obtain

$$\left(\int_{\mathbb{H}^+}|\nabla_H u|^2 d\xi-\int_{\mathbb{H}^+}\frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi\right)^{\frac{1}{2}} \geq C\left(\int_{\mathbb{H}^+}|u|^{2^*}d\xi\right)^{\frac{1}{2^*}},$$

showing (3.9). □

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